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# Conformal Invariance of Maxwell's Equations with Magnetic Charges

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It is shown that Maxwell's equations when there exist isolated magnetic charges are invariant under conformal transformations.

A number of equations in physics seem to be invariant not only under Poincaré transformations but also under conformal transformations. Conformal invariance was introduced into physics by Cunningham (1909) and Bateman (1909) shortly after Einstein's first paper on special relativity; they showed that Maxwell's equations are invariant in form under the 15-parameter conformal group, which has the 10-parameter Poincaré group as a subgroup. Only equations describing massless particles are conformal invariant unless we transform masses in a conformally covariant way (Schouten and Haantjes, 1940). The role of conformal invariance in field theory has been investigated by several people (Wess, 1960; Mack and Salam, 1969).

In recent years Dirac's hypothesis (Dirac, 1931; 1949) of the existence of isolated magnetic charge—the Dirac monopole—has stimulated extensive theoretical studies (Strazhev and Tomil'chik, 1973). Also, several attempts were made to detect such a charge experimentally (Strazhev and Tomil'chik, 1973). It was announced recently that Dirac's monopole was found experimentally (Price et al., 1975), but the interpretation as a monopole of what was seen in the experiment was not met without objections (Fleischer and Walker, 1975). The existence of magnetic charge gives Maxwell's equations a symmetric form between electric and magnetic quantities. Also, it provides a nice explanation of the quantization of the electric charge. There are no theoretical reasons which exclude the existence of the magnetic charge, and therefore several consequences of its existence have been examined.

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In the present paper we shall show that Maxwell's equations with magnetic charges are invariant in form under conformal transformations. This is something that we expected since the photon remains massless. Our proof is relatively simple, self-contained, and independent of the form of Maxwell's equations in the Riemannian space–time of general relativity. Of course the proof of invariance includes as a special case the case of absence of magnetic charges.

Let $x^\mu$, $\mu = 0, 1, \ldots, 3$ be the components of a point in a flat space–time, and let us consider the point transformations (active transformations)

$$x'^\mu = x'^\mu(x^0, x^1, \ldots, x^3) \tag{1.1}$$

which determine the components of the point $x'^\mu$ in a coordinate system, when the components of the point $x^\mu$ are known in the same coordinate system. If the line elements $ds(x')$ and $ds(x)$ at the points $x'$ and $x$ are connected by the relation

$$ds(x') = \left\| \frac{\partial x'}{\partial x} \right\|^{1/4} ds(x) \tag{1.2}$$

where $\|\partial x'/\partial x\|$ is the absolute value of the Jacobian determinant of (1.1), the transformations (1.1) are called conformal transformations. It is obvious from equation (1.2) that the metric tensor $g_{\mu\nu}$ does not behave under conformal transformations as a true tensor, but it behaves as a tensor density

$$g'_{\mu\nu} = \left\| \frac{\partial x'}{\partial x} \right\|^{1/2} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma} \tag{1.3}$$

The transformation law of the tensor $g^{\mu\nu}$, which is defined by the relation $g^{\mu\nu} g_{\nu\rho} = \delta_\rho{}^\mu$, is easily obtained from equation (1.3).

It has been shown that a set of fields $\Psi(x)$ which belong to a linear representation of the inhomogeneous Lorentz group behave under conformal transformations as (Isham et al., 1970)

$$\Psi'(x') = \left\| \frac{\partial x'}{\partial x} \right\|^{l^\Psi/4} D\left( \left\| \frac{\partial x'}{\partial x} \right\|^{-1/4} \frac{\partial x'^\mu}{\partial x^\nu} \right) \Psi(x) \tag{1.4}$$

where $D(\Lambda_\nu{}^\mu)$ is the Lorentz transformation matrix of the field $\Psi(x)$ and $l^\Psi$ is a Lorentz scalar called conformal weight of the field $\Psi(x)$. For example,

$$B'^\mu(x') = \left\| \frac{\partial x'}{\partial x} \right\|^{(l^B - 1)/4} \frac{\partial x'^\mu}{\partial x^\nu} B^\nu(x) \tag{1.5}$$

Also, the operators $\partial_\mu = \partial/\partial x^\mu$ and $\partial^\mu = g^{\mu\nu}\partial_\nu$ transform as follows:

$$\partial'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \tag{1.6}$$

$$\partial'^{\mu} = \left\|\frac{\partial x'}{\partial x}\right\|^{1/2} \frac{\partial x'^{\mu}}{\partial x^{\rho}}\, \partial^{\rho} \tag{1.7}$$

In the absence of magnetic charges the electromagnetic interactions are introduced by the minimal substitution $\partial^{\mu} \to \partial^{\mu} - ieA^{\mu}(x)$, where $A^{\mu}(x)$ is the electromagnetic four-vector potential. From equations (1.5) and (1.7) we see that in order to achieve invariance of the action we have to take $l^{A} = -1$. This choice combined with equation (1.7) implies that in the absence of magnetic charges the electromagnetic field strength $\bar{f}^{\mu\nu}(x) = \partial^{\mu}A^{\nu}(x) - \partial^{\nu}A^{\mu}(x)$ transforms as a second-rank tensor density of conformal weight $l^{\bar{f}} = -2$. An invariant action integral must have zero conformal weight, which implies that the Lagrangian density $\mathscr{L}$ must have $l^{\mathscr{L}} = -4$. Then the interaction Lagrangian density $\mathscr{L}_{\text{int}} = eA_{\mu}(x)\bar{J}^{\mu}(x)$, where $\bar{J}^{\mu}(x)$ is the electromagnetic current in the absence of magnetic charges, implies that the current density $\bar{J}^{\mu}(x)$ has conformal weight $l^{\bar{J}} = -3$.

Maxwell's equations in the presence of magnetic charges take the form

$$\partial_{\mu} f^{\mu\nu}(x) = J^{\nu}(x) \tag{1.8}$$

$$\partial_{\mu}^{*} f^{\mu\nu}(x) = {}^{*}J^{\nu}(x) \tag{1.9}$$

where

$${}^{*}f^{\mu\nu}(x) = \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} f_{\rho\sigma}(x) \tag{1.10}$$

$f_{\rho\sigma}(x)$ is the electromagnetic field strength (which now cannot be written in the form $\partial_{\rho}A_{\sigma} - \partial_{\sigma}A_{\rho}$), $J^{\nu}(x)$, ${}^{*}J^{\nu}(x)$ are the electric and magnetic currents, respectively, and $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi–Civita symbol.

Conformal invariance of Maxwell's equations means that if we express them in the coordinates $x'^{\nu}$, which satisfy equations (1.1) and (1.2), they get the same form, i.e.,

$$\partial'_{\mu} f'^{\mu\nu}(x') = J'^{\nu}(x') \tag{1.11}$$

$$\partial'^{*}_{\mu} f'^{\mu\nu}(x') = {}^{*}J^{\nu}(x') \tag{1.12}$$

where

$$f'^{\mu\nu}(x') = \left\|\frac{\partial x'}{\partial x}\right\|^{-1} \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}}\, f^{\rho\sigma}(x) \tag{1.13}$$

$${}^{*}f'^{\mu\nu}(x') = \left|\frac{\partial x'}{\partial x}\right|^{-1} \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}}\, {}^{*}f^{\rho\sigma}(x) \tag{1.14}$$

$$J'^{\nu}(x') = \left\|\frac{\partial x'}{\partial x}\right\|^{-1} \frac{\partial x'^{\nu}}{\partial x^{\sigma}}\, J^{\sigma}(x) \tag{1.15}$$

$${}^{*}J'^{\nu}(x') = \left|\frac{\partial x'}{\partial x}\right|^{-1} \frac{\partial x'^{\nu}}{\partial x^{\sigma}}\, {}^{*}J^{\sigma}(x) \tag{1.16}$$

Equations (1.13) and (1.15) indicate that the quantities $f^{\mu\nu}(x)$ and $J^{\nu}(x)$ transform under conformal transformations as second-rank tensor and as vector, respectively, of conformal weights $l^f = -2$ and $l^J = -3$, as in the case of an absence of magnetic charges. This should be the case because the presence of magnetic charges should not affect the transformation properties of $f^{\mu\nu}$ and $J^{\nu}$. The magnetic current should transform like the electric current except that in the transformation formula the Jacobian determinant will appear and not its absolute value. This current is an axial vector and it should transform as in equation (1.16). The expression (1.14) can be derived from equations (1.10) and (1.13). Indeed, using equation (1.3) we get

$$f'_{\mu\nu}(x') = g'_{\mu\rho}(x')g'_{\nu\sigma}(x')f'^{\rho\sigma}(x') = \frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}} f_{\rho\sigma} \tag{1.17}$$

Using the relation

$$\epsilon'^{\mu\nu\rho\sigma} = \left|\frac{\partial x'}{\partial x}\right|^{-1} \frac{\partial x'^{\mu}}{\partial x^{\lambda}}\frac{\partial x'^{\nu}}{\partial x^{\tau}}\frac{\partial x'^{\rho}}{\partial x^{\kappa}}\frac{\partial x'^{\sigma}}{\partial x^{\eta}}\epsilon^{\lambda\tau\kappa\eta} = \epsilon^{\mu\nu\rho\sigma} \tag{1.18}$$

we get from equations (1.10), (1.17), and (1.18)

$$*f'^{\mu\nu}(x') = \tfrac{1}{2}\epsilon'^{\mu\nu\rho\sigma}f'_{\rho\sigma}(x') = \left|\frac{\partial x'}{\partial x}\right|^{-1} \frac{\partial x'^{\mu}}{\partial x^{\lambda}}\frac{\partial x'^{\nu}}{\partial x^{\tau}} *f^{\lambda\tau}(x) \tag{1.19}$$

which is identical to equation (1.14). The above equation indicates that the expression $*f^{\mu\nu}(x)$ transforms as pseudotensor density, as expected from its definition.

To prove equation (1.12) we need the relation

$$\left|\frac{\partial x'}{\partial x}\right|^{-1} \partial^{\lambda}\left|\frac{\partial x'}{\partial x}\right| = \frac{\partial^2 x'^{\kappa}}{\partial x^{\lambda}\partial x^{\tau}}\frac{\partial x^{\tau}}{\partial x'^{\kappa}} \tag{1.20}$$

Indeed, using equations (1.6), (1.9), (1.14), (1.16), (1.20), and the antisymmetry of the tensor $f^{\rho\sigma}(x)$, we get

$$\begin{aligned} \partial'_{\mu}*f'^{\mu\nu}(x') = \left|\frac{\partial x'}{\partial x}\right|^{-1}\Bigg[ & -\frac{\partial x^{\lambda}}{\partial x'^{\mu}}\left|\frac{\partial x'}{\partial x}\right|^{-1}\left(\partial^{\lambda}\left|\frac{\partial x'}{\partial x}\right|\right)\frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}} f^{\rho\sigma}(x) \\ & + \frac{\partial x^{\lambda}}{\partial x'^{\mu}}\frac{\partial^2 x'^{\mu}}{\partial x^{\lambda}\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}} f^{\rho\sigma}(x) + \frac{\partial^2 x'^{\nu}}{\partial x^{\rho}\partial x^{\sigma}} f^{\rho\sigma}(x) + \frac{\partial x'^{\nu}}{\partial x^{\sigma}}\partial_{\rho} f^{\rho\sigma}(x)\Bigg] = *J'^{\nu}(x') \end{aligned} \tag{1.21}$$

which is equation (1.12). The proof of equation (1.11) proceeds in exactly the same way, since equation (1.20) also holds if we replace the determinant by its absolute value. Therefore Maxwell's equations in the presence of magnetic charges are invariant under conformal transformations.

From equations (1.13) and (1.14) we easily find that the expressions

$f'^{\mu\nu}(x')$ and $*f'^{\mu\nu}(x')$ are antisymmetric in the indices $\mu$ and $\nu$. Therefore from equations (1.11) and (1.12) we obtain

$$\partial'_\nu J'^\nu(x') = \partial'_\nu {}^*J'^\nu(x') = 0 \tag{1.22}$$

which means that conformal transformations do not affect the conservation laws of the electric and the magnetic charges.

The transformation properties of the electric current and the magnetic current under conformal transformations, i.e., equations (1.15) and (1.16), were introduced independently of one another. It is possible, however, to show that these laws are not independent, but that they are derived from the transformation of a single quantity. To understand this we write Maxwell's equations with magnetic charges in a manifest "dyality" invariant form (Han and Biedenharn, 1971). This is done by expressing the electromagnetic field tensor as a function of the antisymmetric second-rank Hertz tensor $\mathscr{H}^{\mu\nu}$, which consists of the electric and the magnetic Hertz vectors. These vectors are called sometimes Hertz potentials. The source currents are expressed as four divergences of an antisymmetric second-rank tensor field $\mathscr{S}^{\mu\nu}$ called the source tensor (Han and Biedenharn, 1971) or stream potential (Nisbet, 1955; Laporte and Uhlenbeck, 1931). The Hertz tensor $\mathscr{H}^{\mu\nu}$ is given by

$$\mathscr{H}^{\mu\nu} = \begin{pmatrix} 0 & \pi_x & \pi_y & \pi_z \\ & 0 & \Sigma_z & -\Sigma_y \\ & & 0 & \Sigma_x \\ & & & 0 \end{pmatrix} \qquad \mathscr{H}^{\mu\nu} = -\mathscr{H}^{\nu\mu} \tag{1.23}$$

where $\boldsymbol{\pi}$ is the electric Hertz vector and $\boldsymbol{\Sigma}$ the magnetic Hertz vector. It can be shown that the manifest dyality invariant form of Maxwell's equations is (Han and Biedenharn, 1971)

$$\Box \mathscr{H}^{\mu\nu} = \mathscr{S}^{\mu\nu} \tag{1.24}$$

The electric current $J^\nu$ is given by

$$J^\nu = \partial_\mu \mathscr{S}^{\mu\nu} \tag{1.25}$$

while the magnetic current $*J^\nu$ is the "dyality" transform of the electric current, given by (Han and Biedenharn, 1971)

$$*J^\nu = \partial_\mu {}^*\mathscr{S}^{\mu\nu} \tag{1.26}$$

where

$$*\mathscr{S}^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\mathscr{S}_{\rho\sigma} \tag{1.27}$$

Maxwell's equations (1.8) and (1.9) are easily obtained from equations (1.24)–(1.27) if we make the identification

$$f^{\mu\nu} = \Box \mathscr{H}^{\mu\nu} \tag{1.28}$$

It is obvious from equations (1.25) and (1.26) that the transformation law under conformal transformations of the electric and the magnetic currents are completely specified once the transformation law of the stream potential is given. We assume that $\mathscr{S}^{\mu\nu}$ transforms under conformal transformations as a second-rank tensor density of conformal weight $l^s = -2$, i.e., like $f^{\mu\nu}$ [equation (1.13)]. Then proceeding as before we can show that the electric and the magnetic currents $J^\nu$ and $*J^\nu$, defined by equations (1.25) and (1.26), respectively, transform like those of equations (1.15) and (1.16).

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